Problem Set IV, Selected Solutions

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October 6, 2014

Problem 0.1. Prove that if there exists a linear map on V whose null space and range are both finite dimensional, then V is finite dimensional.

Proof. Suppose there exists a linear map $T : V \to W$ with W some vector space such that $\operatorname{null} T$ and $\operatorname{range} T$ are finite dimensional. Then there are vectors $u_1, \dots, u_m \in V$ and $w_1, \dots, w_n \in \operatorname{range} T$ such that $(u_1, \dots, u_m)$ is the span of $\operatorname{null} T$ and $(w_1, \dots, w_n)$ spans $\operatorname{range} T$. Clearly, since each $w_i \in \operatorname{range} T$, there is $v_i \in V$ such that $w_i = Tv_i$.

Take some $v \in V$. Then $T(v) \in \operatorname{range} T$, so there are scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ such that

$$T(v) = \alpha_1 w_1 + \cdots + \alpha_n w_n = T(\alpha_1 v_1 + \cdots + \alpha_n v_n). \tag{0.1}$$

This implies that $v - \alpha_1 v_1 - \cdots - \alpha_n v_n \in \operatorname{null} T$. Thus, there are scalars $\beta_1, \dots, \beta_m \in \mathbb{F}$ such that

$$v - \alpha_1 v_1 - \cdots - \alpha_n v_n = \beta_1 u_1 + \cdots \beta_m u_m \Rightarrow v = \beta_1 u_1 + \cdots + \beta_m u_m + \alpha_1 v_1 + \cdots + \alpha_n v_n. \tag{0.2}$$

Thus, $(u_1, \dots, u_m, v_1, \dots, v_n)$ spans V, so V is finite dimensional.

Remark 0.2. Some people proposed a proof by Rank-Nullity. However, that theorem assumes what we want to prove, namely that V is finite dimensional!

□

Problem 0.3. Suppose that W is finite dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective iff there exists $S \in \mathcal{L}(V, W)$ such that ST is the identity map on V.

Proof. Let's handle the easier direction first. Suppose there exists $S \in \mathcal{L}(V, W)$ such that ST is the identity on V. Take $v_1, v_2 \in V$ such that $T(v_1) = T(v_2)$. Then we have

$$v_1 = (ST)(v_1) = S(T(v_1)) = S(T(v_2)) = (ST)(v_2) = v_2. \tag{0.3}$$

Thus T is injective.

Now suppose that T is injective. Define a map $\rho : \operatorname{range} T \to V$, given explicitly by $T(v) \mapsto v$. This map is linear on $\operatorname{range} T$. Furthermore, this map is well-defined by the injectivity of T, since we can write each element in $\operatorname{range} T$ as Tv in only one way.

By a previous homework exercise, we can extend ρ to a linear map $S \in \mathcal{L}(V, W)$. For some $v \in V$,

$$(ST)v = S(Tv) = \rho(Tv) = v, \tag{0.4}$$

so $ST = \mathrm{id}_V$. □